MATHEMATICS SPECIALIST

MAWA Semester 2 (Unit 1 & Unit 2) Examination 2017

Calculator - assumed

Marking Key

Section One: Calculator-free (100 Marks)

Question 10(a)

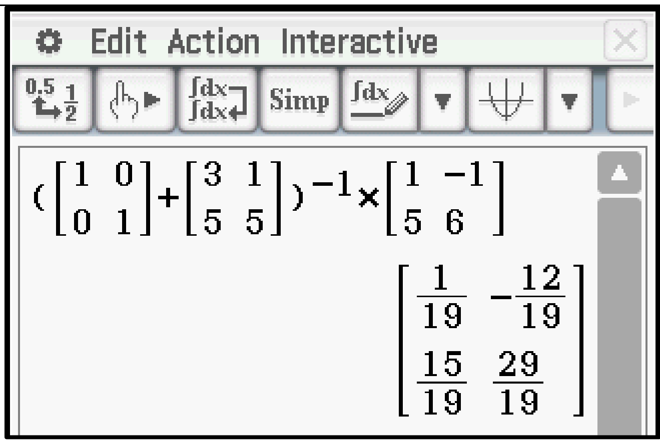
Solution

$C = A - BC$

$C + BC = A$

$(I + B)C = A$

$C = (I + B)^{-1} A$



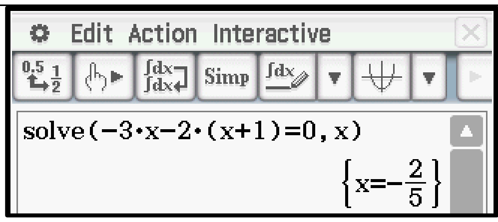
Marking key/mathematical behaviours	Marks
• rearranges the matrix equation to isolate A	1
• uses the identity matrix	1
• pre –multiplies by inverse matrix	1
• solves for matrix A	1

Question 10(b)

Solution

$\text{Det} = 3\times -x - 2\times(x+1) = 0$

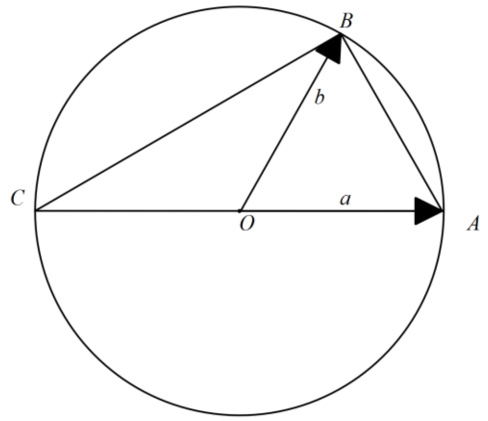
$\Leftrightarrow -5x = 2, \quad x = \frac{-2}{5}$



Marking key/mathematical behaviours	Marks
• obtains an expression for determinant	1
• equates determinant to zero	1
• solves for x	1

Question 11(a)

Solution



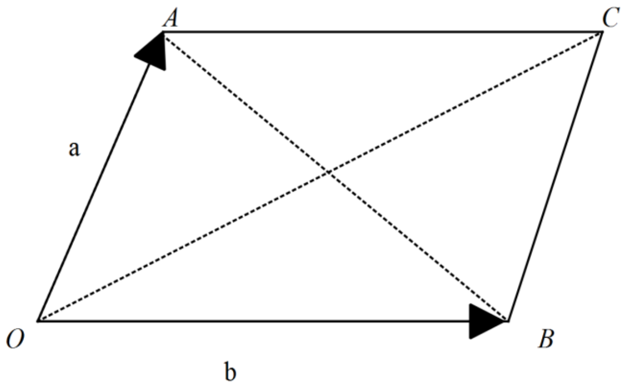
Prove that $\angle CBA = \frac{\pi}{2}$

$$\overrightarrow{CB} \bullet \overrightarrow{BA} = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$$
$$= \mathbf{a} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b}$$
$$= |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0 \text{ since } a \text{ and } b \text{ are radii}$$
$$\therefore |\mathbf{a}||\mathbf{b}| \times Cos \ \angle CBA = 0$$
$$\Leftrightarrow Cos \ \angle CBA = 0, |\mathbf{a}| = |\mathbf{b}| \neq 0$$
$$\Rightarrow \angle CBA = \frac{\pi}{2}$$

Marking key/mathematical behaviours	Marks
• Uses dot product • Determines expression for $\overrightarrow{CB}$ • Determines expression for $\overrightarrow{BA}$ • Shows that dot product equals zero • Summarizes result	1 1 1 1 1

Question 11(a)

Solution



RTP: $|OC|^2 + |AB|^2 = |AO|^2 + |OB|^2 + |BC|^2 + |CA|^2$

$$\begin{aligned}|OC|^2 + |AB|^2 &= \overrightarrow{OC}\bullet\overrightarrow{OC} + \overrightarrow{AB}\bullet\overrightarrow{AB}\\ &= (\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b}) + (\mathbf{b}-\mathbf{a})\bullet(\mathbf{b}-\mathbf{a})\\ &= \mathbf{a}\bullet\mathbf{a} + 2\mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{b} - 2\mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{b}\\ &= \mathbf{a.a} + \mathbf{b.b} + \mathbf{a.a} + \mathbf{b.b} \text{ since } |BC| = |AO| \text{ and } |CA| = |OB|\end{aligned}$$

$|OC|^2 + |AB|^2 = |AO|^2 + |OB|^2 + |BC|^2 + |CA|^2$

Marking key/mathematical behaviours	Marks
• expresses statement as an equation.	1
• expresses diagonals as vectors in terms of **a** and **b**	1
• uses dot product for magnitude squared	1
• states that $\lvert BC\rvert = \lvert AO\rvert$ and $\lvert CA\rvert = \lvert OB\rvert$	1
• shows both sides are equal	1

Question 12(a)(i)

Solution

$\underset{\sim}{a}.\underset{\sim}{a} = |\underset{\sim}{a}|^2 = 7^2$

Marking key/mathematical behaviours	Marks
• defines and uses scalar (dot) product	1

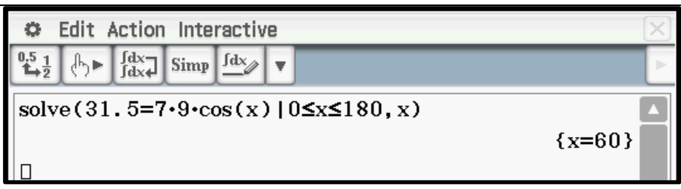
Question 12(a)(ii)

Solution

$\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| Cos(x)$

$\Leftrightarrow 31.5 = 7 \times 9 \times Cos(x)$

$\Rightarrow x = 60^\circ$



Marking key/mathematical behaviours	Marks
• uses dot product formula	1
• solves for angle	1

Question 12(a)(iii)

Solution

Projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ $= |\underset{\sim}{a}| \cos\theta \times \hat{\underset{\sim}{b}}$

Magnitude of projection $= |\underset{\sim}{a}| \cos\theta \times 1$

$= 7 \times \frac{1}{2} = 3.5$

Marking key/mathematical behaviours	Marks
• uses correct formula	1
• solves for scalar projection	1

Question 12(b)(i)

Solution

$\underset{\sim}{d} = 21i - 7j \quad \underset{\sim}{w} = 2i - \mu j \quad \underset{\sim}{v} = \lambda i + 15j$

$\underset{\sim}{d} \bullet \underset{\sim}{w} = 0 \Leftrightarrow \begin{pmatrix} 21 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -\mu \end{pmatrix} = 0$

$\Leftrightarrow 42 + 7\mu = 0$

$\Leftrightarrow \mu = -6$

Marking key/mathematical behaviours	Marks
• applies dot product = zero	1
• determines dot product	1
• solves unknown	1

Question 12(b)(ii)

Solution

$$\begin{pmatrix} 21 \\ -7 \end{pmatrix} = \alpha \begin{pmatrix} \lambda \\ 15 \end{pmatrix}$$

$15\alpha = -7$ and $\alpha\lambda = 21$

$$\Rightarrow \alpha = \frac{-7}{15} \text{ and } \lambda = \frac{21}{\alpha} = 15\frac{21}{-7} = -45$$

Marking key/mathematical behaviours	Marks
• uses scalar multiple	1
• sets up equations for λ and α	1
• solves for λ	1

Question 13 (a)

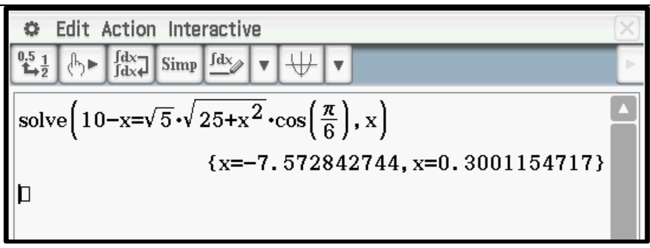
Solution

$$\underset{\sim}{c} \bullet \underset{\sim}{d} = |\underset{\sim}{c}| \times |\underset{\sim}{d}| Cos(\theta)$$

$$\Leftrightarrow Cos\left(\frac{\pi}{6}\right) = \frac{(2i + j) \bullet (5i - xj)}{|\underset{\sim}{c}| \times |\underset{\sim}{d}|}$$

$$\Leftrightarrow 10 - x = \sqrt{5} \times \sqrt{25 + x^2} \times Cos\left(\frac{\pi}{6}\right)$$

$x =$ -7.57 or $x = 0.30$



Marking key/mathematical behaviours	Marks
• uses dot product formula	1
• sets up equation with x	1
• solves for one x value	1
• solves for both to two decimal places	1

Question 13 (b)

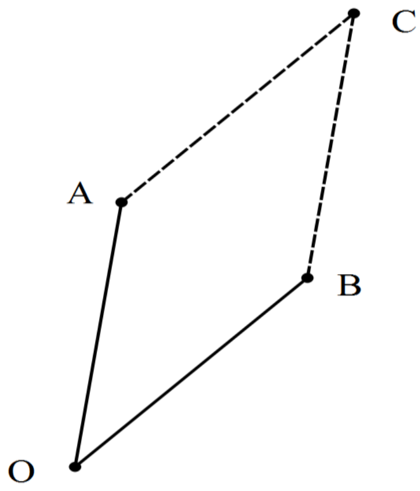
Solution

A sketch of OACB may help.

$\overrightarrow{OA}=\begin{pmatrix}1\\7\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}5\\5\end{pmatrix}$

$\Rightarrow \overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OB}=\begin{pmatrix}6\\12\end{pmatrix}$

$\therefore C$ is $(6,12)$



Marking key/mathematical behaviours	Marks
• adds position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ • states the correct coordinates	1 1

Question 13 (c)

Solution

$\overrightarrow{OA}=\begin{pmatrix}1\\7\end{pmatrix}$ $\overrightarrow{OB}=\begin{pmatrix}5\\5\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}6\\12\end{pmatrix}$

Consider diagonals $\overrightarrow{AB}$ and $\overrightarrow{OC}$

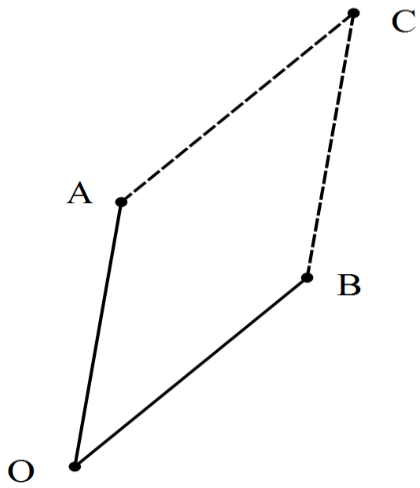
$\overrightarrow{AB}=\begin{pmatrix}5\\5\end{pmatrix}-\begin{pmatrix}1\\7\end{pmatrix}=\begin{pmatrix}4\\-2\end{pmatrix}$

$\overrightarrow{AB}\bullet\overrightarrow{OC}=\begin{pmatrix}4\\-2\end{pmatrix}\bullet\begin{pmatrix}6\\12\end{pmatrix}=4\times6+(-2)\times12$

$=24-24=0$

$\Rightarrow \overrightarrow{AB}\perp\overrightarrow{OC}$ $\Rightarrow$ the diagonals of $OACD$ are perpendicular

$\Rightarrow OACD$ is a rhombus



Marking key/mathematical behaviours	Marks
• subtracts vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to get second diagonal • shows the dot product of the diagonals = 0 • concludes that the diagonals are perpendicular • concludes that the given figure is a rhombus	1 1 1 1

Question 14(a)(i)

Solution 52-25=27 81-18=63	
Marking key/mathematical behaviours	Marks
• one example • two examples	1 1

Question 14(a)(ii)

Solution The difference is always a multiple of three	
Marking key/mathematical behaviours	Marks
States the correct result	1

Question 14(a)(iii)

Solution The result is always true $ab = 10a + b$ $ba = 10b + a$ $difference = \lvert 10(a-b) + b - a \rvert$ $= \lvert 9(a-b) \rvert$ $= 3\lvert 3(a-b) \rvert$	
Marking key/mathematical behaviours	Marks
• states that result is always true • expresses two-digit numbers in place value algebraically • shows that difference is a multiple of three	1 1 1

Question 14(b)

Solution Assume that $x =$ smallest rational number greater than zero exists. The value of $\frac{x}{2}$ is smaller than x but still rational $0 \le \frac{x}{2} \le x$, contradiction. $\therefore$ assumption is wrong and that there is no smallest rational number greater than zero.	
Marking key/mathematical behaviours	Marks
assumes smallest rational number greater than zero, exists derives an illogical result summarizes that opposite is true	1 1 1

Question 15

Solution

$n(A)=\frac{100}{2}=50$ $n(B)=\frac{100}{3}=33$ $n(C)=\frac{100}{5}=20$

$n(B\cap C)=\frac{100}{15}=6$ $n(A\cap B)=\frac{50}{3}=17$ $n(A\cap C)=\frac{50}{5}=10$

$n(A\cap B\cap C)=\frac{50}{15}=3$

= Hence

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(B\cap C)-n(A\cap B)-n(A\cap C)+n(A\cap B\cap C)$$
$$=50+33+20-6-17-10+3=73$$
$$n\overline{(A\cup B\cup C)}=100-73=27$$

Therefore numbers which are odd and/or divisible by 3 or 5 = $50+33+20-17-10-6+3=73$

Therefore numbers which are not odd nor divisible by 3 nor 5 = $100-73=27$

Marking key/mathematical behaviours	Marks
• determines correctly the seven subsets	4
• correctly uses the inclusion-exclusion principle to determine $n(A\cup B\cup C)$	1
• correctly determines $n\overline{(A\cup B\cup C)}$	1

Question 16(a)

Solution

$$4\sin\theta\cos^2\frac{\theta}{2}$$
$$=2\sin\theta\ 2\cos^2\frac{\theta}{2}$$
$$=2\sin\theta(\cos\theta+1)$$
$$=2\sin\theta\cos\theta+2\sin\theta$$
$$=\sin 2\theta+2\sin\theta$$
$$\therefore A=1, B=2$$

Marking key/mathematical behaviours	Marks
• expresses as $2\sin\theta\ 2\cos^2\frac{\theta}{2}$	1
• applies $2\cos^2\frac{\theta}{2}=\cos\theta+1$	1
• expands brackets and recognises $\sin 2\theta=2\sin\theta\cos\theta$	1
• determines correct values for A and B	1

Question 16(b)

Solution

$\sin 5x - \sin x = \sin 2x \qquad 0 \le x \le \pi$

$\Leftrightarrow 2\cos\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right) = \sin 2x$ converting to a product

$\Leftrightarrow 2\cos 3x \sin 2x = \sin 2x$

$\Leftrightarrow \sin 2x(2\cos 3x - 1) = 0$

$\Rightarrow \sin 2x = 0$ or $(2\cos 3x - 1) = 0 \qquad 0 \le 2x \le 2\pi$ and $0 \le 3x \le 3\pi$

$\Rightarrow 2x = 0, \pi, 2\pi$ or $3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\Rightarrow x = 0, \frac{\pi}{2}, \pi$ or $x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$\therefore\ x = 0, \frac{\pi}{9}, \frac{\pi}{2}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi$

Marking key/mathematical behaviours	Marks
• replaces $\sin 5x - \sin x$ with $2\cos 3x \sin 2x$	1
• rearranges equation to obtain $\sin 2x = 0$ and $(2\cos 3x - 1) = 0$	1
• solves both equations for x within $0 \le x \le \pi$	1
• states all the correct solutions (note: use of degrees is not acceptable – domain is given in radians)	1

Question 17(a)

Solution

$y = \frac{1}{2}\sin 2x - 2$

Marking key/mathematical behaviours	Marks
• determines correct vertical dilation value of $\frac{1}{2}$ and horizontal dilation value of 2	1
• determines correct vertical translation value of -2 and equation	1

Question 17(b)

Solution

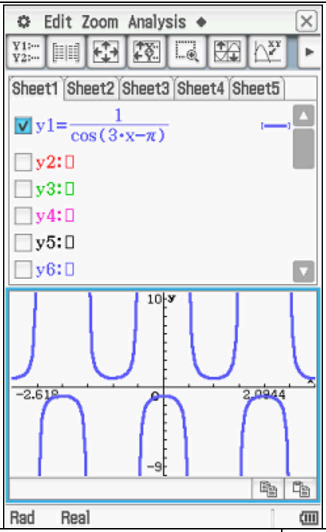
For graph $y=\dfrac{1}{\cos(3x-\pi)}$, $0\le x\le \pi$

asymptotes occur at when $\cos(3x-\pi)=0$

$3x-\pi=-\dfrac{\pi}{2},\dfrac{\pi}{2},\dfrac{3\pi}{2}$ $\qquad -\pi\le(3x-\pi)\le 2\pi$

Vertical asymptotes are

$x=\dfrac{\pi}{6}, x=\dfrac{\pi}{2}, x=\dfrac{5\pi}{6}$



Marking key/mathematical behaviours	Marks
• recognises graph does not exist when $\cos(3x-\pi)=0$	1
• states the correct equations of the vertical asymptotes	1

Question 17(c)

Solution

$$\tan x = \frac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}$$

Let $x = \frac{5\pi}{4}$

$$\Leftrightarrow \tan\frac{5\pi}{4} = \frac{2\tan\frac{5\pi}{8}}{1-\tan^2\frac{5\pi}{8}}$$

$$\Leftrightarrow 1\left(1-\tan^2\frac{5\pi}{8}\right) = 2\tan\frac{5\pi}{8}$$

$$\Leftrightarrow \tan^2\frac{5\pi}{8} + 2\tan\frac{5\pi}{8} - 1 = 0$$

$$\Leftrightarrow \tan\frac{5\pi}{8} = \frac{-2\pm\sqrt{4-4(1)(-1)}}{2(1)}$$

$$\tan\frac{5\pi}{8} = \frac{-2\pm\sqrt{8}}{2}$$

$$\tan\frac{5\pi}{8} = -1+\sqrt{2} \text{ or } \tan\frac{5\pi}{8} = -1-\sqrt{2}$$

Since $0 < \frac{5\pi}{8} < \pi$ then $\tan\frac{5\pi}{8} = -1-\sqrt{2}$

Marking key/mathematical behaviours	Marks
• recognises to use $x = \frac{5\pi}{4}$	1
• expresses $\tan\frac{5\pi}{4} = \frac{2\tan\frac{5\pi}{8}}{1-\tan^2\frac{5\pi}{8}}$	1
• rearranges equation and uses the quadratic formula	1
• uses the quadratic formula correctly to solve for $\tan\frac{5\pi}{8}$ for 2 possible values	1
• recognises that $\tan\frac{5\pi}{8}$ is negative and states the correct value	1

Question 18(a)

Solution	
In a circle, chords of equal length subtend equal angles at the centre. Converse: In a circle, equal angles at the centre are subtended by equal chords at the circumference.	
Marking key/mathematical behaviours	Marks
• states a correct converse	1

Question 18(b)

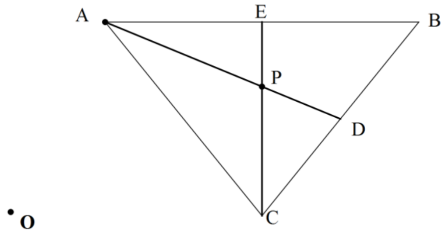
Solution	
Statement: $P \Rightarrow Q$ Contrapositive: not $Q \Rightarrow$ not P (i) Hence, if an angle is not a right angle then the angle at the circumference is not subtended by a semi-circle. (ii) Correct. An angle which is > 90 degrees at the circumference will be subtended by an arc that has a reflex angle > 180 degrees at the centre and hence the arc will be greater than a semi-circle. Similarly, an angle which is < 90 degrees at the circumference will be subtended by an arc that has an angle < 180 degrees at the centre and hence the subtending arc will be smaller than a semi-circle.	
Marking key/mathematical behaviours	Marks
• provides a correct contrapositive statement • indicates that the contrapositive statement is correct and provides a reasonable explanation • Provides a good explanation covering both cases	1 1 1

Question 18(c)

Solution	
Any $a > b$ where $b < 0 < a$ e.g. $2 > -3 \Rightarrow \frac{1}{2} < -\frac{1}{3}$ is false	
Marking key/mathematical behaviours	Marks
• provides a correct counter example	1

Question 19(a)

Solution



$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$

$\Rightarrow \overrightarrow{AE} = \dfrac{\mathbf{b} - \mathbf{a}}{2}$

$\therefore \overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AE}$

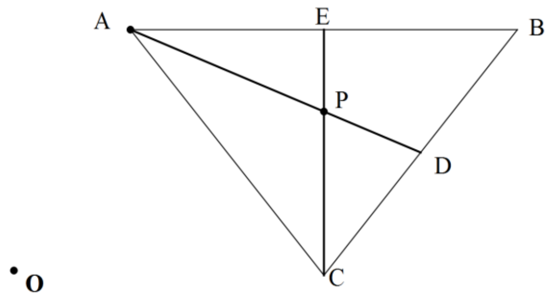
$= -\mathbf{c} + \mathbf{a} + \dfrac{\mathbf{b} - \mathbf{a}}{2}$

$= \dfrac{1}{2}\mathbf{a} + \dfrac{1}{2}\mathbf{b} - \mathbf{c}$

Marking key/mathematical behaviours	Marks
• determines $\overrightarrow{AE}$ in terms of the given position vectors	1
• determines a correct expression for $\overrightarrow{CE}$	1

Question 19(b)

Solution



$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = \overrightarrow{AO} + \overrightarrow{OC} + \overrightarrow{CP}$

$= -\mathbf{a} + \mathbf{c} + \dfrac{2}{3}(\overrightarrow{CE})$

$= \mathbf{c} - \mathbf{a} + \dfrac{2}{3}(\dfrac{1}{2}\mathbf{a} + \dfrac{1}{2}\mathbf{b} - \mathbf{c})$ using result from part (a)

$= -\dfrac{2\mathbf{a}}{3} + \dfrac{1}{3}\mathbf{b} + \dfrac{1}{3}\mathbf{c}$

$= \dfrac{1}{3}(\mathbf{b} + \mathbf{c} - 2\mathbf{a})$

Marking key/mathematical behaviours	Marks
• determines $\overrightarrow{AP}$ in terms of the given position vectors and $(\overrightarrow{CE})$	1
• uses the expression for $\overrightarrow{CE}$ from part (a)	1
• simplifies to arrive at the correct result	1

Question 20

Solution

Rotation 1 is given by $R_A = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Rotation 2 is given by $R_B = \begin{bmatrix} \cos(-B) & -\sin(-B) \\ \sin(-B) & \cos(-B) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Combined rotation is given by $R_{BA} = \begin{bmatrix} \cos B & \sin B \\ -\sin B & \cos B \end{bmatrix}\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} \cos B\cos A + \sin B\sin A & \cos A\sin B - \sin A\cos B \\ \sin A\cos B - \cos A\sin B & \cos B\cos A + \sin B\sin A \end{bmatrix}$$

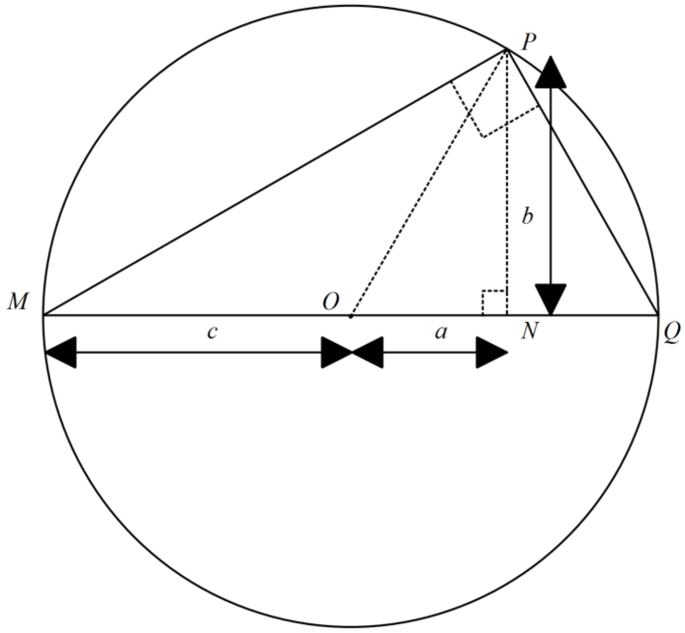
Combined rotation is equivalent a rotation of $(A-B) = \begin{bmatrix} \cos(A-B) & -\sin(A-B) \\ \sin(A-B) & \cos(A-B) \end{bmatrix}$

Hence $\cos(A-B) = \cos B\cos A + \sin B\sin A$

Marking key/mathematical behaviours	Marks
• uses two rotation matrices	1
• multiplies matrices in correct order	1
• simplifies trig expressions of $-B$	1
• calculates the correct element of the matrix product	1
• equates product to single rotation of A-B	1

Question 21(a)

Solution



In ΔMPN and ΔPQN

$\angle PMN + \angle MPN = 90^\circ$ (angle sum in Δ with right angle)

$\angle MPN + \angle NPQ = 90^\circ$ (given $\angle MPQ = 90^\circ$)

$\therefore \angle PMN = \angle NPQ$

$\therefore \angle MNP = \angle PNQ$

Therefore triangles similar due to AA test

Marking key/mathematical behaviours	Marks
• uses both Δ's have right angles	1
• shows that both Δ' have two equal acute angles which together with AA test implies they are similar	1

Question 21(a)

Solution

$\frac{NQ}{PN} = \frac{PN}{MN}$ similar Δ's

$\Rightarrow \frac{c-a}{b} = \frac{b}{c+a}$

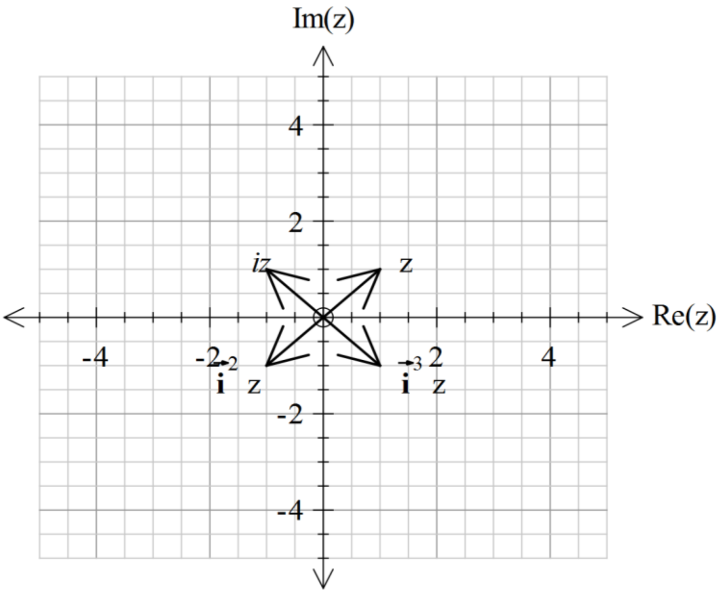
$\Rightarrow c^2 - a^2 = b^2$ by cross multiplication

$\Rightarrow c^2 = a^2 + b^2$

Marking key/mathematical behaviours	Marks
• identifies corresponding sides are in proportion	1
• states side lengths in terms of $a, b \& c$	1
• shows the required result	1

Question 22(a)

Solution



Marking key/mathematical behaviours	Marks
• Plots all 4 vectors (1 mark each)	4

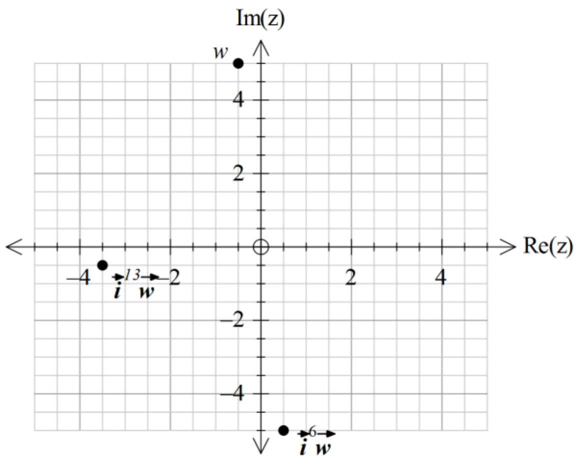
Question 22(b)

Solution
No change to modulus, argument increases by anticlockwise 90 degrees

Marking key/mathematical behaviours	Marks
• States no change to modulus • States the correct direction anticlockwise by • States the correct magnitude of 90 degrees	1 1 1

Question 22(c)

Solution



Marking key/mathematical behaviours	Marks
• Plots w correctly • Plots $i^6 w$ and $i^{13} w$ correctly	1 2